An optimization model for finding nonnegative polynomials and its application to some filter design problems

Thanh Hieu Le[1] **and Marc Van Barel**[2]

[1]Department of Mathematics, Quynhon University

[2]Department of Computer Science, KU Leuven

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- Non-negative, sos(m) polynomials are used in the formulation of several problems:
  - to formulate filter design problems
  - to approximate Lyapunov functions for ODEs, PDIs , etc
- nonnegativity: NP-hard
- sos-representation: solvable in polynomial time
- real nonnegative polynomial $\approx$ real sos-polynomial
- sos(m) polynomials are defined by positive semidefinite matrices
- positive semidefinite matrices allows us to use SDPs

- Study an optimization model of finding nonnegative polynomials defining on certain sets
- Express coefficients of such polynomials as linear functions in entries of corresponding Gram matrices
- Reformulate the main model as a conic linear program
- Apply to some low-pass filter design problems

$$f = \sum_{i=1}^{m} |f_i|^2 = \sum_{i=1}^{\pi(f)\leq m} |g_i|^2$$

$$\mathbb{R}[x]_{n,d} = \{h \in \mathbb{R}[x_1, \ldots, x_n] : \deg(h) \leq d\}$$

$$\mathbb{C}[z]_{n,d} = \{h \in \mathbb{C}[z_1, \ldots, z_n] : \deg(h) \leq d\}$$

sos-polynomials:

$$f \in \Sigma(n, 2d) \Longleftrightarrow f_i \in \mathbb{R}[x]_{n,d}$$

sosm-polynomials:

$$f \in \Sigma^{\Im}(n, d) \Longleftrightarrow f_i \in \mathbb{C}[z]_{n,d}$$

f is sosrf-polynomial $\overset{\text{def}}{\Longleftrightarrow}$ $f_i = \dfrac{u_i}{v_i}, u_i, v_i \in \mathbb{R}[x]$

One variable: $\Big\{f:\ f\geq 0 \text{ on } \mathbb{R}\Big\} = \Big\{f:\ f \text{ is a sum of at most } 2 \text{ squares}\Big\}.$

$$\begin{aligned} 1+2x^2+x^4 &= (1+x^2)^2, \\ 1+x^2+\ldots+x^{2014} &= (x^2+1)\prod_{k=1}^{503}\left[(x^2-\cos\tfrac{k\pi}{504})^2+\sin^2\tfrac{k\pi}{504}\right] \end{aligned}$$

$$\boxed{(a^2+b^2)(c^2+d^2) = (ac+bd)^2+(ad-bc)^2}$$

Several variables: Motzkin polynomial $f^{\mathrm{M}} = 1-3x^2y^2+x^2y^4+x^4y^2$ is

- non-negative on $\mathbb{R}^2$
- a sum of squares of rational functions
- <u>not</u> a sum of squares of real polynomials

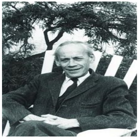
$\longrightarrow \Sigma(n,2d) \subsetneq P(n,2d)$ in general!

- Hilbert [1888]:
  - univariate polynomials, $n = 1, d \in \mathbb{N}$
  - quadratic polynomials, $n \in \mathbb{N}, d = 2$
  - bivariate quartics, $n = 2, d = 4$

D. Hilbert (1862-1943)

- Artin [1927]: (the 17th problem of Hilbert)

$$\begin{aligned} & \{f \in \mathbb{R}[x]_{n,2d} : \ f \geq 0 \text{ on } \mathbb{R}^n\} \\ = \ & \{f \in \mathbb{R}[x]_{n,2d} : \ f \text{ is sosrf } \}. \end{aligned}$$



E. Artin (1898-1962)

- Lasserre [2004]:

$$\Big\{f \in \mathbb{R}[x] : \ f \text{ is sos on } \mathbb{R}^n\Big\} \subsetneq_{\text{dense}} \Big\{f \in \mathbb{R}[x] : \ f \geq 0 \text{ on } \mathbb{R}^n\Big\}$$

One variable:

$g(z) = \sum_{k=-d}^{d} a_k z^k, \quad a_{-k} = \bar{a}_k \in \mathbb{C}, \quad z \in \mathbb{T} \triangleq \{z \in \mathbb{C} : \ |z| = 1\}.$

$$\Big\{g : \ g \geq 0 \text{ on } \mathbb{T}\Big\} = \Big\{g : \ g(z) = |h(z)|^2, h \in \mathbb{C}[z]\Big\}.$$

- $1 + |z|^2 = 2, \quad z \in \mathbb{T}$
- $1 + |(2+\imath) + (1-\imath)z|^2 = \left| \frac{3}{\sqrt{4+\sqrt{6}}} + \frac{\imath}{\sqrt{4+\sqrt{6}}} + \imath\sqrt{4+\sqrt{6}}z \right|^2, \quad z \in \mathbb{T}$

Several variables:

- Rudin [1963] : $g \ \geq \ 0$ on $\mathbb{T}^n \not\Longrightarrow g = \sum_i |g_i|^2$
- Dritschel [2004] : $g \ > \ 0$ on $\mathbb{T}^n \Longrightarrow g = \sum_i |g_i|^2$

- $y^4 + (\sqrt{2}xy)^2 + x^4 = \begin{bmatrix} y^2 \\ xy \\ x^2 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y^2 \\ xy \\ x^2 \end{bmatrix}$
  - ▶ real monomials: $y^2, \ xy, \ x^2, \ x^2y^2, \ y^4, \ x^4.$

- $z \in \mathbb{T}$:

$$\begin{aligned} 1 + |(2+\imath) + (1-\imath)z|^2 &= 1 + \overline{[2+\imath+(1-\imath)z]}[2+\imath+(1-\imath)z] \\ &= \begin{bmatrix} 1 \\ z \end{bmatrix}^H \overline{\begin{bmatrix} 1 & 2+\imath \\ 0 & 1-\imath \end{bmatrix}} \begin{bmatrix} 1 & 2+\imath \\ 0 & 1-\imath \end{bmatrix}^T \begin{bmatrix} 1 \\ z \end{bmatrix} \\ &= (1+3\imath)\bar{z} + 8 + (1-3\imath)z \end{aligned}$$

  - ▶ complex monomials: $1, \quad z$
  - ▶ complex Laurent monomials: $\bar{z} \quad 1 \quad z$

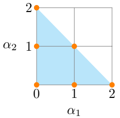
- $\mathbf{v}_d(w)$ = column vector of monomials $w^\alpha \triangleq w_1^{\alpha_1} \dots w_n^{\alpha_n}$, $\alpha \in \mathbb{N}^n$ of degree $\leq d$:

$$\sum_{i=1}^{n} \alpha_i \leq d;$$

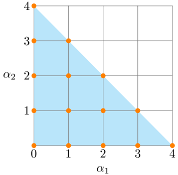
- Sets of exponents α :

$\Omega(n,d)$	$\Gamma(n,d)$	$\Gamma^{\Im}(n,d)$
$\alpha : \sum_{i=1}^n \alpha_i \leq d$	$\Omega(n,d) + \Omega(n,d)$ $= \Omega(n,2d)$	$\Omega(n,d) - \Omega(n,d)$
w^α, w^β	$w^\alpha . w^\beta = w^{\alpha+\beta}$	$\overline{w}^\alpha . w^\beta = w^{\beta-\alpha}$

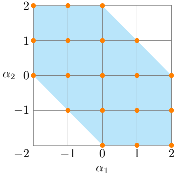
$$\Omega(2,2) = \{(\alpha_1, \alpha_2) \in \mathbb{N}^2 : \ \alpha_1 + \alpha_2 \le d = 2\};$$
$$\Gamma(2,2) = \{(\gamma_1, \gamma_2) \in \mathbb{N}^2 : \ \gamma_1 + \gamma_2 \le 2d = 4\};$$
$$\Gamma^{\Im}(2,2) = \Omega(2,2) - \Omega(2,2) \qquad .$$



$\#\Omega(2,2) = 6$



$\#\Gamma(2,2) = 15$



$\#\Gamma^{\Im}(2,2) = 19$

$\#\Omega(n,d)$	$\#\Gamma(n,d)$	$\#\Gamma^{\Im}(n,d)$
$\binom{n+d}{n}$	$\binom{n+2d}{n}$	NA

$$e \triangleq \binom{n+d}{n}, \quad a \triangleq \binom{n+2d}{n}, \quad a^{\Im} = \#\Gamma^{\Im}(n,d).$$

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & \\
 & Ax \preceq b \\
 & Fx = g,
\end{array}
$$

where $A \in \mathbb{R}^{m\times n}, \quad F \in \mathbb{R}^{p\times n}$

Given $f(w) = \sum_\gamma f_\gamma w^\gamma \in \mathbb{F}[w](\mathbb{R}[x]$ or $\mathbb{L}[z])$.

$$\begin{aligned} f(w) &= \sum_{k=1}^{r} |g_k(w)|^2 = \mathbf{v}_d(w)^H(\overline{G}G^T)\mathbf{v}_d(w) \\ &\Updownarrow \\ f_\gamma &= \sum_{\substack{\beta+\alpha=\gamma \\ (\beta-\alpha=\gamma)}} \Big(\sum_{k=1}^{r} \bar{g}_{k\alpha} g_{k\beta}\Big), \forall\gamma \end{aligned}$$

⇝ sos-polynomials: a equations, er real variables $g_{k\alpha}$.

⇝ sosm-polynomials: $\lfloor \frac{a^{\Im}}{2} \rfloor + 1$ equations, er complex variables $g_{k\alpha}$.

⇝ $A = \overline{G}G^T$ is called Gram matrix; G is called Cholesky matrix of g.

- Artin [1927] solved Hilbert's 17th problem [1900]

$$f \geq 0 \text{ on } \mathbb{R}^n \quad \Longrightarrow \quad \exists q : qf = \sum_i p_i^2, \quad p_i \in \mathbb{R}[x].$$

- Pólya [1928] and Reznick [1995]: For f homogeneous

$$f > 0 \text{ on } \mathbb{R}^n \setminus \{0\} \quad \Longrightarrow \quad f_r \triangleq \left(\sum_{i=1}^n x_i^2\right)^r .f \quad \text{is } \underline{\text{sos}} \text{ for some } r \in \mathbb{N}.$$

Given $p(x) = \sum_{i=0}^{2d} p_i x^i \in \mathbb{R}[x]_{1,2d}$.

- $p \geq 0$ on $\mathbb{R}$ $\iff$ $\exists P \in \mathbb{S}^e_+ : p_i = \mathrm{Trace}(H_i^{(d+1)} P), \ \forall i = 0, \ldots, 2d,$
- $H_i^{(d+1)}$ is the Hankel matrix of order $d+1$ with the first column is the ith identity vector $e_i \in \mathbb{R}^{d_1}$.

Given $p(x) = \sum_{i=0}^{2d} p_i x^i \in \mathbb{R}[x]_{1,2d}$.

- $p \geq 0$ on $[a, b] \subset \mathbb{R}$ if and only if

$$p(x) = \begin{cases} p_1(x)^2 + (x-a)(b-x)p_2(x)^2, \deg p_2 + 1 = \deg p_1 = d_1, & \text{if } d = 2d_1, \\ (x-a)p_1(x)^2 + (b-x)p_2(x)^2, \deg p_1 = \deg p_2 = d_1, & \text{if } d = 2d_1 + 1, \end{cases}$$

p_1, p_2 nonnegative on $\mathbb{R}$.

- $d = 2d_1$:

$$\begin{pmatrix} p_0^0 \\ p_1^0 \\ p_2^0 \\ \vdots \\ p_{d-2}^0 \\ p_{d-1}^0 \\ p_d^0 \end{pmatrix} = \underbrace{\begin{pmatrix} p_0^1 & 0 & 0 & p_0^2 \\ p_1^1 & 0 & p_0^2 & p_2^2 \\ p_2^1 & p_0^2 & p_1^2 & p_3^2 \\ \vdots & \vdots & \vdots & \vdots \\ p_{d-2}^1 & p_{d-4}^2 & p_{d-3}^2 & p_{d-2}^2 \\ p_{d-1}^1 & p_{d-3}^2 & p_{d-2}^2 & 0 \\ p_d^1 & p_{d-2}^2 & 0 & 0 \end{pmatrix}}_{\triangleq L_1} \begin{pmatrix} 1 \\ -1 \\ b+a \\ -ab \end{pmatrix}, \tag{1}$$

- $d = 2d_1 + 1$:

$$\begin{pmatrix} p_0^0 \\ p_1^0 \\ p_2^0 \\ \vdots \\ p_{d-2}^0 \\ p_{d-1}^0 \\ p_d^0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 0 & p_0^2 & p_0^1 \\ p_0^1 & p_0^2 & p_1^2 & p_1^1 \\ \vdots & \vdots & \vdots & \vdots \\ p_{d-1}^1 & p_{d-1}^2 & p_d^2 & p_d^1 \\ p_d^1 & p_d^2 & 0 & 0 \end{pmatrix}}_{\triangleq L_2} \begin{pmatrix} 1 \\ -1 \\ b \\ -a \end{pmatrix}, \tag{2}$$

Given $p(x) = \sum_{i=0}^{2d} p_i x^i \in \mathbb{R}[x]_{1,2d}$.

- $p \geq 0$ on $[0, \infty) \subset \mathbb{R}$ if and only if

$$p(x) = p_1(x)^2 + x p_2(x)^2,$$
$$\deg(p_1) = \lfloor \frac{d}{2} \rfloor, \ \deg(p_2) = \lfloor \frac{d-1}{2} \rfloor,$$
$$p_1, p_2 \geq 0 \text{ on } \mathbb{R}.$$

- d is odd:

$$\begin{pmatrix} p_0^0 \\ p_1^0 \\ \vdots \\ p_{d-1}^0 \\ p_d^0 \end{pmatrix} = \underbrace{\begin{pmatrix} p_0^1 & 0 \\ p_1^1 & p_0^2 \\ \vdots & \vdots \\ p_{2d_1}^1 & p_{2d_1}^2 \\ 0 & p_{2d_1+1}^2 \end{pmatrix}}_{\triangleq L_3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{3}$$

- d id even:

$$\begin{pmatrix} p_0^0 \\ p_1^0 \\ \vdots \\ p_{d-1}^0 \\ p_d^0 \end{pmatrix} = \underbrace{\begin{pmatrix} p_0^1 & 0 \\ p_1^1 & p_0^2 \\ \vdots & \vdots \\ p_{d-1}^1 & p_{d-2}^2 \\ p_d & 0 \end{pmatrix}}_{\triangleq L_4} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4}$$

Given $p(x) = \sum_{\alpha\in\Omega(2,d)} p_\alpha x^\alpha \in \mathbb{R}[x]_{2,d}$.

- $p(x_1,x_2) \geq 0$ on $[0,1]\times\mathbb{R} \Longleftrightarrow p((b-a)x_1 + a, x_2) \geq 0$ on $[a,b]\times\mathbb{R}$.

- Marshall [2009]: $p \geq 0$ on $[0,1] \subset \mathbb{R}$ if and only if

$$p(x_1,x_2) = g(x_1,x_2) + x_1(1-x_1)h(x_1,x_2),$$
$$g = \sum_i g_i^2, h = \sum_j h_j^2 \in \Sigma(2,2k).$$

- Assume
$g_i = \sum_{\alpha\in\Omega(2,k)} g_{i\alpha}x^\alpha,\ h_j = \sum_{\alpha\in\Omega(2,k)} h_{j\alpha}x^\alpha$

Assume

$$\begin{aligned}
g(x_1,x_2) &= \sum_{\sigma=(\sigma_1,\sigma_2)\in\Omega(2,2k)} g_\sigma x_1^{\sigma_1}x_2^{\sigma_2},\\
h(x_1,x_2) &= \sum_{\tau=(\tau_1,\tau_2)\in\Omega(2,2k)} h_\tau x_1^{\tau_1}x_2^{\tau_2},\\
g_\sigma &= \sum_{\alpha_1+\beta_1=\sigma}\Big(\sum_{i=1}^{r} g_{i\alpha_1}g_{i\beta_1}\Big),\\
h_\tau &= \sum_{\alpha+\beta=\tau}\Big(\sum_{j=1}^{s} h_{j\alpha}h_{j\beta}\Big),
\end{aligned}$$

$$\begin{aligned}
p_\mu &= g_\mu + h_\gamma - h_\omega,\\
\gamma &= (\gamma_1,\gamma_2), \omega = (\omega_1,\omega_2),\\
\gamma_1 + 1 &= \omega_1 + 2 = \mu_1,
\end{aligned}$$

Given $g(z) = \sum_{k=-d}^{d} g_k z^k, \quad g_{-k} = \bar{g}_k, g_k = a_k - \imath b_k.$

$$g(z) = \frac{1}{2}\left(\sum_{k=0}^{d}(a_k + \imath b_k)z^{-k} + \sum_{k=0}^{d}(a_k - \imath b_k)z^k\right)$$
$$= \mathrm{Re}\left(\sum_{k=0}^{d} g_k z^k\right) = a_0 + \sum_{k=1}^{d}(a_k \cos k\theta + \imath b_k \sin k\theta),$$
$$z = e^{\imath\theta}, \theta \in [-\pi, \pi].$$

$$g \geq 0 \text{ on } \mathbb{T} \iff \exists A \in \mathbb{H}_+^e : \quad a_k + \imath b_k = \operatorname{Trace}(T_k^{(d+1)} A), \quad k = 0, 1, \ldots, d,$$

where $T_0^{(d+1)} = I_{d+1}$ and

$$T_k^{(d+1)} = \begin{pmatrix} 0 & \ldots & \ldots & \ldots & 0 \\ \vdots & \ddots & & & \vdots \\ 2 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 2 & \ldots & 0 \end{pmatrix}, k = 1, ..., d.$$

Given $p(z) = \sum_{k=-d}^{d} p_k z^k, \;\; p_{-k} = \bar{p}_k, g_k = a_k - \imath b_k.$

$$\mathbb{T}_{uv} = \{z \in \mathbb{T} : \; \arg(u) \leq \arg(z) \leq \arg(v)\}.$$

- $\omega_u = \arg(u)$.
- $p \geq 0$ on $\mathbb{T}_{uv}$ if and only if
  $\exists p_1(z), p_2(z) \in \Sigma(1, d), \deg(p_1) = d, \deg(p_2) = d - 1 :$

$$p(z) = p_1(z) + \left(e^{-\imath \frac{\omega_v + \omega_u}{2}} z + e^{\imath \frac{\omega_v + \omega_u}{2}} z^{-1} - 2\cos\frac{\omega_v - \omega_u}{2}\right) p_2(z).$$

$$\begin{pmatrix} p_0^0 \\ p_1^0 \\ p_2^0 \\ \vdots \\ p_{d-2}^0 \\ p_{d-1}^0 \\ p_d^0 \end{pmatrix} = \underbrace{\begin{pmatrix} p_0^1 & p_0^2 & 0 & p_0^2 \\ p_1^1 & p_1^2 & p_0^2 & p_2^2 \\ p_2^1 & p_2^2 & p_1^2 & p_3^2 \\ \vdots & \vdots & \vdots & \vdots \\ p_{d-2}^1 & p_{d-2}^2 & p_{d-3}^2 & p_{d-1}^2 \\ p_{d-1}^1 & p_{d-1}^2 & p_{d-2}^2 & 0 \\ p_d^1 & 0 & p_{d-1}^2 & 0 \end{pmatrix}}_{\triangleq M} \begin{pmatrix} 1 \\ -2\cos(\frac{\omega_v - \omega_u}{2}) \\ e^{-\imath\frac{\omega_v+\omega_u}{2}} \\ e^{\imath\frac{\omega_v+\omega_u}{2}} \end{pmatrix}, \qquad (5)$$

where p_t^0, p_t^1, p_t^2 are coefficients of p, p_1, p_2, respectively.

$$\begin{aligned} &\text{minimize} \quad \delta \in (a,b) \subset \mathbb{R}_+ \\ &\text{subject to} \\ &\qquad (p_1,\ldots,p_m) \in \mathcal{K}, \\ &\qquad q_i \triangleq \sum_{i=1}^{m} a_{ij}(\delta) p_j \in \mathcal{K}_i, \quad i = 1,\ldots,\mu \end{aligned} \tag{6}$$

- $\mathcal{K}$: set of $m-$tuples of all complex Laurent/real polynomials
- $\mathcal{K}_i$: cone of sos(m)-polynomials, or univariate polynomials ≥ 0 on $[a,b], [0,+\infty), \mathbb{T}_{uv}$
- $A(\delta)$: $\mu \times m$ matrix with $a_{ij}(\delta) \in \mathbb{R}[\delta]$.

For each $\delta \in (a, b)$.
Consider each polynomial as a column vector of its coefficients

Find $\quad (p_1, \ldots, p_m) \in \mathcal{K}$

subject to

$$q_i \triangleq \sum_{i=1}^{m} a_{ij}(\delta) p_j \in \mathcal{K}_i,$$
$$i = 1, \ldots, \mu$$

Find $\quad \mathbf{p} = [\mathbf{p}_1^T, \ldots, \mathbf{p}_m^T]^T$

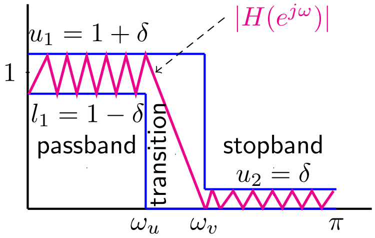
subject to

$$\begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_1 \\ \vdots \\ \mathbf{q}_\mu \end{bmatrix} = \begin{bmatrix} a_{11}(\delta)I & \ldots & a_{1m}(\delta)I \\ a_{21}(\delta)I & \ldots & a_{2m}(\delta)I \\ \vdots & \ddots & \vdots \\ a_{\mu}(\delta)I & \ldots & a_{\mu m}(\delta)I \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_m \end{bmatrix} \quad (7$$

$\mathbf{q}_i \leftrightsquigarrow$ Gram matrices

- Can be solved by combining a bisection rule on δ & a conic linear program solving the feasibility problem! (see later)

- IIR low-pass filter: $H(e^{\imath\omega}) = \frac{a_0+a_1z+\ldots+a_dz^d}{b_0+b_1z+\ldots+b_dz^d}, \quad z = e^{\imath\omega}, \quad \omega \in [-\pi,\pi]$.
- FIR low-pass filter: $H(e^{\imath\omega}) = \sum_{i=1}^{d} h_i z^i, \quad z = e^{\imath\omega}, \quad \omega \in [-\pi,\pi]$.



minimize δ
subject to

$$\begin{array}{ll} |H(e^{\imath\omega})|^2 \geq l_1^2, & \omega \in [0,\omega_u], \\ |H(e^{\imath\omega})|^2 \leq u_1^2, & \omega \in [0,\omega_v], \\ |H(e^{\imath\omega})|^2 \leq \delta^2, & \omega \in [\omega_v,\pi], \\ |H(e^{j\omega})| \geq 0, & \omega \in [0,\pi]. \end{array}$$

Consider a class of discrete-time systems

$$\sum_{k=0}^{d} a_k w[t-k] = \sum_{k=0}^{d} b_k y[t-k],$$

$\{a_k\}_{k=0}^{d}, \{b_k\}_{k=0}^{d}$: real numbers
$y[i]$: the input signal at discrete time i.
$w[i]$: the output signal at discrete time i.
The corresponding frequency response is the function

$$H(e^{j\theta}) = \frac{\sum_{k=0}^{d} a_k e^{-jk\theta}}{\sum_{k=0}^{d} b_k e^{-jk\theta}}.$$

Sufficient to consider $H(e^{j\theta})$ on $[0, \pi]$ instead of $(-\pi, \pi]$.

Different design problems can be considered. Here are three standard ones:

1. minimize the passband ripple, given a stopband attenuation:

$$\text{minimize}(u_1 - l_1); d, \delta \text{ are fixed}$$

2. minimize the stopband attenuation:

$$\text{minimize}(\delta); d, l_1, u_1 \text{ are fixed}$$

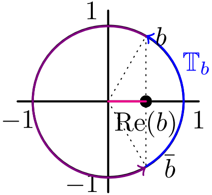
3. minimize the degree d of the filter:

$$\text{minimize}(d); \delta, l_1, u_1 \text{ are fixed.}$$

$$|H(z)|^2 = \frac{p_1(z)}{p_2(z)}, \quad z = e^{\imath\omega}, \quad p_1, p_2 \in \mathbb{L}[z], \; p_1, p_2 \geq 0 \text{ on } \mathbb{T}.$$

$0 < \omega_a < \omega_b < \pi, \;\; \omega \in [-\pi, \pi]:$

- $|\omega| \in [0, \omega_a] \iff z \in \mathbb{T}_a$
- $|\omega| \in [0, \omega_b] \iff z \in \mathbb{T}_b$
- $|\omega| \in [\omega_b, \pi] \iff z \in \mathbb{T}_{\bar{b}}$



minimize δ
subject to

$$\begin{aligned} &|H(e^{\imath\omega})|^2 \geq l_1^2, && \omega \in [0, \omega_u], \\ &|H(e^{\imath\omega})|^2 \leq u_1^2, && \omega \in [0, \omega_v], \\ &|H(e^{\imath\omega})|^2 \leq \delta^2, && \omega \in [\omega_v, \pi], \\ &|H(e^{j\omega})| \geq 0, && \omega \in [0, \pi]. \end{aligned}$$

minimize δ
subject to

$$\begin{aligned} p_1, p_2 \in \;& \mathbb{L}[z], && \\ q_1(z) \triangleq \;& p_1(z) \geq 0, && \forall z \in \mathbb{T}, \\ q_2(z) \triangleq \;& p_1(z) - l_1^2 p_2(z) \geq 0, && \forall z \in \mathbb{T}_a, \\ q_3(z) \triangleq \;& u_1^2 p_2(z) - p_1(z) \geq 0, && \forall z \in \mathbb{T}_b, \\ q_4(z) \triangleq \;& u_2^2 p_2(z) - p_1(z) \geq 0, && \forall z \in \mathbb{T}_{\bar{b}}. \end{aligned}$$

$$
\begin{array}{rcll}
p_1, p_2 & \in & \mathbb{L}[z], & \\
q_1(z) & \triangleq & p_1(z) \geq 0, & \forall z \in \mathbb{T}, \\
q_2(z) & \triangleq & p_1(z) - l_1^2 p_2(z) \geq 0, & \forall z \in \mathbb{T}_a, \\
q_3(z) & \triangleq & u_1^2 p_2(z) - p_1(z) \geq 0, & \forall z \in \mathbb{T}_b, \\
q_4(z) & \triangleq & u_2^2 p_2(z) - p_1(z) \geq 0, & \forall z \in \mathbb{T}_{\bar{b}}.
\end{array}
$$

$q_1 \geq 0$ on $\mathbb{T}$ $\quad\Longleftrightarrow\quad \exists X \in \mathbb{H}_+^{d+1} : \; q_{1k} = \ell_1(X),$

$i = 2, 3, 4 :$

$q_i \geq 0$ on $\mathbb{T}_u$ $(T_{\bar{u}})$ $\quad\Longleftrightarrow\quad \exists (X_1, X_2) \in \mathbb{H}_+^{d+1} \times \mathbb{H}_+^{d} : \; q_{ik} = \ell_{i2}(X_1, X_2).$

$$\begin{array}{llll} q_1 & \leftrightsquigarrow & Y_1 \in \mathbb{H}^{d+1} & : \quad Y_1 + sI \succeq 0; \\ q_i & \leftrightsquigarrow & (Y_{i1}, Y_{i2}) \in \mathbb{H}^{d+1} \times \mathbb{H}^d & : \quad Y_{i1} + sI, Y_{i2} + sI \succeq 0, \;\; i = 2, 3, 4. \end{array}$$

$$\mathbb{H}^\mu \ni Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1\nu} \\ y_{21} & y_{22} & \cdots & y_{2\nu} \\ \vdots & \vdots & \ddots & \vdots \\ y_{\nu 1} & y_{\nu 2} & \cdots & y_{\nu\nu} \end{bmatrix} \longmapsto \tilde{y} = (y_{11}, y_{12}, y_{1\nu}, y_{22}, \ldots, y_{2\nu}, \ldots, y_{\nu\nu})$$

$$\begin{array}{lllll} \text{Find} & p_1, p_2 & \in & \mathbb{L}[z], & \\ \text{subject to} & q_1(z) & \triangleq & p_1(z) \geq 0, & \forall z \in \mathbb{T}, \\ & q_2(z) & \triangleq & p_1(z) - l_1^2 p_2(z) \geq 0, & \forall z \in \mathbb{T}_a, \\ & q_3(z) & \triangleq & u_1^2 p_2(z) - p_1(z) \geq 0, & \forall z \in \mathbb{T}_b, \\ & q_4(z) & \triangleq & u_2^2 p_2(z) - p_1(z) \geq 0, & \forall z \in \mathbb{T}_{\overline{b}}. \end{array}$$

Given a value $\delta \in (0, 1)$.

$$
\begin{array}{ll}
\text{minimize} & s \\
\text{subject to} & x \in \mathcal{S} \\
& \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -l_1^2 I \\ -I & u_1^2 I \\ -I & u_2^2 I \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}
\end{array} \tag{8}
$$

$$
\begin{aligned}
\mathcal{S} = \quad & \{[s, \tilde{y}_1, \tilde{y}_{21}, \tilde{y}_{22}, \tilde{y}_{31}, \tilde{y}_{32}, \tilde{y}_{41}, \tilde{y}_{42}, \mathbf{p}_1^T, \mathbf{p}_2^T]^T \in \mathbb{R}^{1+\mu} : \\
& Y_1 + sI, Y_{ij} + sI \succeq 0\},
\end{aligned}
$$

$$
\mu = \quad 4\frac{(d+1)(d+2)}{2} + 3\frac{d(d+1)}{2} + 2(d+1).
$$

- wish: $s \leq 0$!

Input. $d, 0 < \omega_a < \omega_b < \pi, \delta_0 \in (0,1)$ such that $s(\delta_0) < 0$, precision $\epsilon > 0$.
Output. A value $\delta \in (0, \delta_0]$ and two cosine polynomials p_1, p_2 which are nonnegative on the unit circle and solve IIR filter problem.

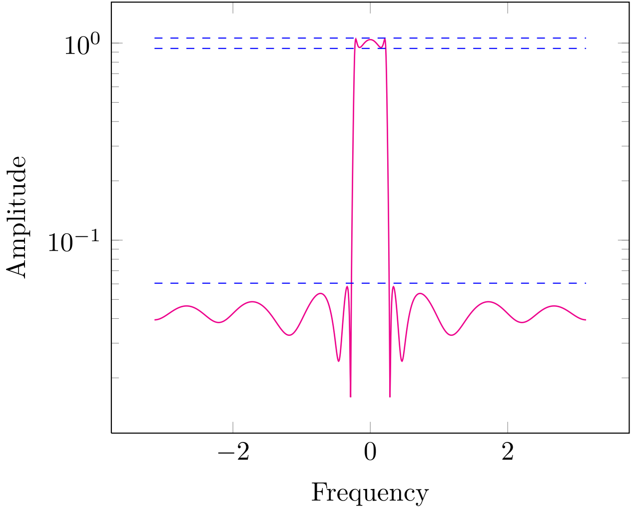
Initialization. Set $\delta_{low}^0 = 0, \delta_{up}^0 = \delta_0$
At iteration $k \geq 0$. Set $\delta^{k+1} = \frac{\delta_{up}^k + \delta_{low}^k}{2}$.

While $(\delta_{up}^k - \delta_{low}^k)/\delta_{up}^k > \epsilon$ do

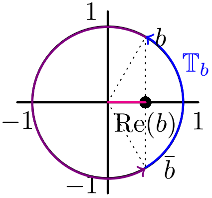
1. Solve the convex optimization problem (8) for $d, \omega_a, \omega_b, \delta^{k+1}$, obtain the optimal value $s^{k+1} = s(\delta^{k+1})$.
2. If $s^{k+1} > 0$ then set $\delta_{low}^{k+1} = \delta^{k+1}, \delta_{up}^{k+1} = \delta_{up}^k$
   Else, set $\delta_{up}^{k+1} = \delta^{k+1}, \delta_{low}^{k+1} = \delta_{low}^k$.
3. Go to Iteration $k+1$.

The value δ is δ^k at which $(\delta_{up}^k - \delta_{low}^k)/\delta_{up}^k \leq \epsilon$ firstly and $s^k < 0$.

$$d = 9, \omega_a = 0.225, \omega_b = 0.275, \delta_{min} = 75 \times 10^{-4}$$



- $z = e^{\imath\omega} = \cos\omega + \imath\sin\omega,\ \omega \in [-\pi, \pi]$
- Complex Laurent polynomial & trigonometric polynomial:
  $p(z) = \sum_{k=-d}^{d} p_k z^k = \sum_{k=0}^{d} (a_k \cos k\omega + \imath b_k \sin k\omega)$
- Cosine polynomial: $b_k = 0, \forall i$
- $x = \cos\omega \in [-1, 1]$
- cosine polynomial $\overset{1-1}{\longleftrightarrow}$ real polynomial on $[-1, 1]$



- cosine polynomial: $p(z) = \sum_{k=0}^{d} a_k \cos k\theta, \theta \in \mathbb{R}$

- $\cos(k\theta) = 2^{k-1}\cos^k\theta + k\sum_{r=1}^{\lfloor k/2 \rfloor} \frac{(-1)^r}{r} \quad \binom{k-r-1}{r-1} 2^{k-2r-1}(\cos\theta)^{k-2r}$

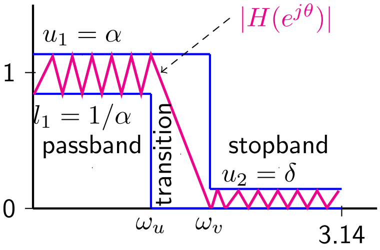
- corresponding polynomial on $[-1,1]$:
  $\hat{p}(x) = \hat{p}_0 + \hat{p}_1 x + \ldots + \hat{p}_d x^d \in \mathbb{R}[x], x \in [-1,1]$

$$\begin{bmatrix} \hat{p}_0 \\ \hat{p}_1 \\ \vdots \\ \hat{p}_d \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 & \ldots & * \\ & 1 & 0 & -3 & \ddots & \vdots \\ & & 2^1 & 0 & \ddots & 0 \\ & & & 2^2 & \ddots & * \\ & & & & \ddots & 0 \\ & & & & & 2^{d-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{bmatrix}$$

$$\begin{array}{lrcl}
\text{minimize } \delta & & & \\
\text{subject to} & p_1, p_2 & \in & \mathbb{R}[x], \\
& q_1(x) & \triangleq & p_1(x) \geq 0, \forall x \in [-1, 1], \\
& q_2(x) & \triangleq & p_1(x) - l_1^2 p_2(x) \geq 0, \forall x \in [Re(a), 1], \\
& q_3(x) & \triangleq & u_1^2 p_2(x) - p_1(x) \geq 0, \forall x \in [Re(b), 1], \\
& q_4(x) & \triangleq & u_2^2 p_2(x) - p_1(x) \geq 0, \forall x \in [-1, Re(b)].
\end{array}$$

$H(e^{\imath\theta}) = \sum_{k=0}^{d} a_k e^{-\imath k\theta}$

$a_k \in \mathbb{R}, \theta \in [0, \pi]$.



minimize δ
subject to

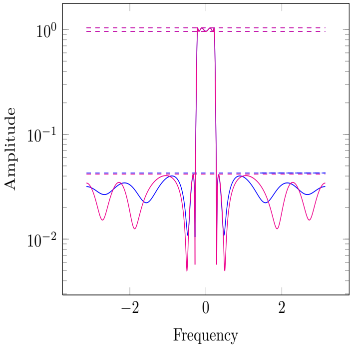
$$\begin{aligned} p &\triangleq |H(z)|^2 \in \Sigma^{\Im}(1,d), \\ q_1(z) &\triangleq p(z) \geq 0, \forall z \in \mathbb{T}, \\ q_2(z) &\triangleq p(z) - l_1^2 \geq 0, \forall z \in \mathbb{T}_a, \\ q_3(z) &\triangleq u_1^2 - p(z) \geq 0, \forall z \in \mathbb{T}_b, \\ q_4(z) &\triangleq u_2^2 - p(z) \geq 0, \forall z \in \mathbb{T}_{\bar{b}}. \end{aligned}$$

IIR filters on complex/real polynomials:

$$d = 9, \;\; \omega_u = 0.225, \;\; \omega_v = 0.275$$

$$\delta_{min} = 0.0417$$

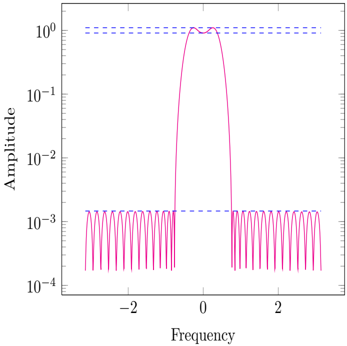
$$\delta_{min} = 0.0428$$



FIR filter on complex polynomials:

$$d = 30, \;\; \omega_u = 12\pi, \;\; \omega_v = 0.24\pi$$

$$\delta_{min} = 0.1465$$



Thank you for your attention!